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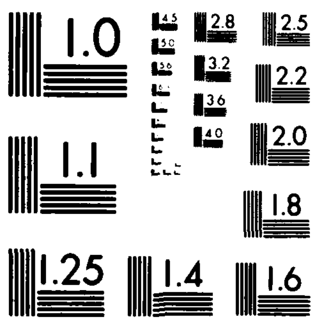
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STABILITY THEORY OF A CONFINED TOROIDAL PLASMA PART II. MODIFIED ENERGY PRINCIPLE AND GROWTH RATE

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ABSTRACT

Based upon the existence and uniqueness of a solution to the linearized Lundquist equations established previously, the modified energy principle for the σ-stability of a confined toroidal plasma is rigorously justified. A variational principle is developed to find the infimum of σ, and an estimate for the maximum growth rate is obtained. The results are also extended to a diffuse pinch and a multiple tori plasma.

AMS(MOS) Subject Classifications: 76W05, 76E25

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SIGNIFICANCE AND EXPLANATION

In this report, we shall justify rigorously the so-called modified energy principle for the σ-stability of a confined toroidal plasma. Intuitively speaking, a plasma is called σ-stable if it does not grow faster than $\exp(\sigma t)$, where $\sigma = 1/\tau$ and τ is some characteristic time needed for fusion. The modified energy principle claims that the plasma is σ-stable if some energy functional is nonnegative and unstable if otherwise. We develop a method to get an upper bound for σ and the maximum growth rate for the plasma is also obtained. Furthermore the results are extended to the case that the plasma fills up the whole conducting shell and that there are several plasma tori in the conducting shell.

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STABILITY THEORY OF A CONFINED TOROIDAL PLASMA
PART II. MODIFIED ENERGY PRINCIPLE AND GROWTH RATE

Peter Laurence* and M. C. Shen†

§1. Introduction.

The MHD stability of a plasma equilibrium confined in a magnetic field is one of the most important topics in controlled thermonuclear research. The approach to this problem is very often based upon the linear energy principle formulated by Berstein et. al. (1957). In the justification of the necessary condition for stability, they assumed that the eigenfunctions of a certain linear operator form a complete orthonormal basis. This assumption may limit the scope of the linear energy principle in applications. Laval et. al. (1965) relaxed this restriction and established a modified energy principle for the so-called σ-stability of a confined plasma, including the linear energy principle as a special case. However, in their derivation the existence of a classical solution to the corresponding system of the linearized Lundquist equations is tacitly assumed, and at present no such a solution is known to exist. In this report we shall state and prove rigorously a precise version of the modified energy principle, for a confined toroidal plasma, which is the main contribution of our work. Our approach relies upon the existence of a generalized solution to the linearized Lundquist equations we established in a previous report, which we shall refer to as Part I.

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The contents of this report are briefly explained as follows. In Section 2, we present the rigorous proof of a precise version of the modified energy principle for σ-stability. A necessary and sufficient condition is given for an equilibrium to be σ-stable. The proof of the sufficiency essentially follows the results obtained in Part I; that of the necessity needs some regularity and embedding theorems by Lions and Magenes (1972). In Section 3, a variational principle is developed to find the infimum of σ for σ-stability. In Section 4, an estimate of the maximum growth rate is obtained. Finally, we extend previous results to the cases of a diffusion pinch and a multiple tori configuration in Section 5.

§2. Necessity and Sufficiency of the Modified Energy Principle.

We begin with the definition of σ-stability of a plasma equilibrium.

Definition An equilibrium is called σ-stable if all solutions $(\xi(t),A(t))$ of the evolutionary variational problem EVP (§6, Part I) have the property that there exist some constant $c > 0$ and σ such that

$$\|\xi\|^2_{2,v} \leq c \exp 2\sigma t \quad \text{on} \quad [0,\infty).$$

We now prove

Theorem 2.1.

An equilibrium subject to the conditions of §3, Part I is σ-stable if and only if

$$a((\xi,A),(\xi,A)) + \sigma^2\|\xi\|^2_{2,\rho} > 0, \qquad \forall(\xi,A) \in W. \tag{2.1}$$

Proof:

To prove the sufficiency, assume that $\Lambda(t) = (\xi(t),A(t))$ is a solution to the EVP with $(\xi(0),A(0)) = \Lambda_0$, $\frac{\partial\xi}{\partial t}(0) = \dot\xi_0$ and prescribed flux $\int_{\Gamma_v} A \cdot n\, ds = \sigma(t)$ (§6, Part I). As in §6, Part I, we associate with the EVP an auxiliary EVP (AEVP) for which we construct the m^{th} Galerkin approximation and obtain (6.16), Part I as follows:

$$\|\dot\xi^n(t)\|^2_{2,\rho} + a((\xi^m,A^m),(\xi^{mn},A^m)) = \|\dot\xi^m(0)\|_{2,\rho} + a(\Lambda^m(0),\Lambda^m(0)).$$

Adding $\sigma^2\|\xi^m\|^2_{2,\rho}$ to both sides and using (2.1) we get

$$\|\dot\xi^m(t)\|^2_{2,\rho} < \|\dot\xi^m(0)\|^2_{2,\rho} + c_1\|\Lambda^m(0)\|^2 + \sigma^2\|\xi^m(t)\|^2_{2,\rho}, \tag{2.2}$$

where c_1 was defined after (6.16), Part I, from which we derive as in passing from (6.20) to (1.22), Part I

$$\|\xi^m(t)\|^2_{2,\rho} < \{\|\xi^m(0)\|^2_{2,\rho} + \frac{b^m}{\sigma^2}\}e^{2\sigma t} - \frac{b^m}{\sigma^2}, \tag{2.3}$$

where

$$b^m = \|\dot\xi^m(0)\|^2_{2,\rho} + c_1\|\Lambda^m(0)\|^2_W .$$

We let

$$b = \|\dot{\xi}(0)\|^2_{2,\rho} + c_1\|\Lambda(0)\|^2_W ,$$

and would like to pass to the limit in (2.3). Since the equilibrium satisfies the assumptions in §3, Part I, and by the coerciveness of $a(\cdot,\cdot)$, there exist λ, $\delta > 0$, (where $\lambda > \sigma^2$) such that

$$a((\xi,\Lambda),(\xi,\Lambda)) + \lambda\|\xi\|^2_{2,\rho} > \delta\|(\xi,\Lambda)\|^2_W .$$

Thus proceeding as in §6, Part I for the existence proof and using the remark in §7, Part I, we have

$$\begin{aligned} &\xi^m(t) \to \xi(t) \quad \text{in} \quad L^2[[0,T] : L^2_\rho(\Omega_p)], \\ &\dot{\xi}^m(t) \to \dot{\xi}(t) \quad \text{in} \quad L^2[[0,T] : L^2_\rho(\Omega_p)]. \end{aligned} \tag{2.4}$$

Also given the construction of our Galerkin approximations, we have

$$\begin{aligned} &\xi^m(0) \to \xi(0) \quad \text{in} \quad L^2_\rho(\Omega_p), \\ &\dot{\xi}^m(0) \to \dot{\xi}(0) \quad \text{in} \quad L^2_\rho(\Omega_p), \\ &\Lambda^m(0) \to \Lambda(0) \quad \text{in} \quad W. \end{aligned} \tag{2.5}$$

For $\tau,s \in [0,T]$, $\tau > s$, we now integrate both sides of (2.3) from s to τ and multiply both sides by $1/(\tau - s)$, to obtain

$$\frac{1}{\tau-s}\int_s^\tau \|\xi^m(t)\|^2_{2,\rho}dt < \frac{1}{\tau-s}\int_s^\tau \{\|\xi^m(0)\|^2_{2,\rho} + \frac{b^m}{\lambda}\}e^{2\sigma t}dt - \frac{1}{\tau-s}\int_s^\tau \frac{b^m}{\lambda}\,dt.$$

We now can pass to the limit on the R.H.S. becuase of (2.4) and on the L.H.S. we use $\|u\| < \lim\inf\|u_n\|$, valid for any weakly convergent sequence in a Banach space, to obtain

$$\frac{1}{\tau-s}\int_s^\tau \|\xi(t)\|^2_{2,\rho}\,dt < \frac{1}{\tau-s}\int_s^\tau \{\|\xi(0)\|^2_{2,\rho} + \frac{b}{\lambda}\}e^{2\sigma t}dt - \frac{1}{\tau-s}\int_s \frac{b}{\lambda}\,dt. \tag{2.6}$$

Noting that $\|\xi(t)\|_{2,\rho} \in L^2[0,T] \quad L^1[0,T]$, we now let $\tau \to s$ on the L.H.S., apply the Lebesque differentiation theorem [Rudin, 1974] there, which states that the derivative of the integral of an L^1 function is the L^1 function, and by explicit integration and differentiation on the R.H.S., we

obtain

$$\|\xi(t)\|^2_{2,\rho} < (\|\xi(0)\|^2_{2,\rho} + \frac{b}{\lambda})e^{2\sigma t} - \frac{b}{\lambda}, \tag{2.7}$$

$$< (\|\xi(0)\|^2_{2,\rho} + \frac{b}{\lambda})e^{2\sigma t}.$$

In addition, if we insert (2.7) into (2.2), we have

$$\|\dot{\xi}^m(t)\|^2_{2,\rho} < \|\dot{\xi}^m(0)\|^2_{2,\rho} + c_1\|\Lambda^m(0)\|^2_W + \lambda\{(\|\xi(0)\|^2_{2,\rho} + \frac{b^m}{\lambda})e^{2\sigma t} - \frac{b^m}{\lambda}\}$$

$$= b^m + \lambda(\|\xi(0)\|^2_{2,\rho} + \frac{b^m}{\lambda})e^{2\sigma t} - b^m.$$

Upon going through the same procedure as between (2.5) and (2.6), we may pass to the limit and obtain

$$\|\dot{\xi}(t)\|^2_{2,\rho} < b + \lambda(\|\xi(0)\|^2_{2,\rho} + \frac{b}{\lambda})e^{2\sigma t} - b$$

$$= \lambda(\|\xi(0)\|^2_{2,\rho} + \frac{b}{\lambda})e^{2\sigma t}. \tag{2.8}$$

This establishes the sufficiency of (2.1).

Next we prove the necessity of the modified energy principle for σ-stability. We seek to show here that

$$a((\check{\xi},\check{A}),(\check{\xi},\check{A})) + \sigma\|\check{\xi}\|^2_{2,\rho} < 0$$

for an element $(\check{\xi},\check{A}) \in W$, implies the existence of a solution of the EVP, $(\xi(t),A(t))$, such that

$$\|\xi\|^2_{2,\rho} > c \exp 2\sigma t.$$

In the proof of this result the essential part is played by certain regularity and embedding theorems of Lions and Magenes (1972), which are then coupled with the original proof of Laval et. al. (1965) to yield the results.

We begin by slightly changing our point of view and noting that $\|(\xi,A)\|_W$ defines a norm on $P_{\Omega_p}(W)$ which we will denote by $\|\xi\|_{W_\xi}$. This is clear since, given $\xi \in \Omega_p$, the boundary value problem

$$\nabla \times \nabla \times A = 0, \quad \nabla \cdot A = 0,$$

$$n \times A = -(n \cdot \xi)B^V \quad \text{on} \quad \Gamma_p,$$

$$n \times A = 0 \quad \text{on} \quad \Gamma_v,$$

$$\int_{\Gamma_v} A \cdot n \, ds = 0, \tag{2.9}$$

determines A uniquely, and the dependence on ξ of A is linear.

Let W_ξ^* Be the dual space of W_ξ. We seek to show that the following formula holds.

$$< \frac{\partial^2}{\partial t^2} \xi(t), \xi(t)>_{W_\xi^* \times W_\xi} = \frac{\partial^2}{\partial t^2} <\xi(t),\xi(t)>_{2,\rho} - \tag{2.10}$$

$$2 <\frac{\partial}{\partial t} \xi(t), \frac{\partial}{\partial t} \xi(t)>_{2,\rho} ,$$

where $<\cdot,\cdot>_{W_\xi^* \times W_\xi}$ denotes the action of the linear functional $\frac{\partial^2}{\partial t^2} \xi(t) \in W_\xi^*$ on $\xi(t) \in W_\xi$, and where $(\xi(t),A(t))$ is a solution to the AEVP. We shall also show that $\frac{\partial}{\partial t} <\xi(t),\xi(t)>_{2,\rho}$ is absolutely continuous on $[0,T]$. Now $a((\xi,A),(\tilde{\xi},\tilde{A}))$ defines a bilinear form on W_ξ, which we shall denote by $a_\xi(\xi,\tilde{\xi})$.

In order to prove (2.9), note that since $(\xi(t),A(t))$ is a solution of AEVP

$$< \frac{\partial^2}{\partial t^2} \xi(t),\tilde{\xi}(t)>_{W_\xi^* \times W_\xi} + a_\xi(\tilde{\xi},\xi) = 0, \quad \forall \tilde{\xi} \in W_\xi. \tag{2.11}$$

We now invoke a theorem of Lions and Magenes (1972). The solution of (2.11) has the property

$$\xi(t) \in C^0([0,T] : W_\xi),$$

$$\frac{\partial \xi}{\partial t} \in C^0([0,T] : H_\xi) ,$$

where H_ξ denotes the closure in $L^2(\Omega_p)$ of W_ξ, previously denoted by $P_{\Omega_p}(H_\xi)$. Furthermore, there exist $\xi^\varepsilon(t) \in C^1([0,T] : W_\xi)$, such that $\frac{\partial}{\partial t} \xi^\varepsilon(t) \in C^1([0,T] : W_\xi)$ and

$$\xi^{\varepsilon}(t) \to \xi(t) \qquad \text{in } C^{0}([0,T] : W_{\xi}), \tag{2.12}$$

$$\frac{\partial\xi}{\partial t}^{\varepsilon}(t) \to \frac{\partial\xi}{\partial t}(t) \quad \text{in } C^{0}([0,T] : H_{\xi}), \tag{2.13}$$

$$\frac{\partial^2\xi^{\varepsilon}}{\partial t^2}(t) \to \frac{\partial^2\xi}{\partial t^2}(t) \quad \text{in } L^{2}([0,T] : W_{\xi}^{*}). \tag{2.14}$$

We shall first assume that formula (2.10) holds for ξ^{ε} and show that it holds for ξ. Integrating (2.10) from 0 to τ, $t \in [0,T]$ we have

$$\int_0^{\tau} 2< \frac{\partial^2}{\partial t^2}\xi^{\varepsilon}(t),\xi^{\varepsilon}(t)>_{W_{\xi}^{*}\times W_{\xi}} = \frac{\partial}{\partial t}<\xi^{\varepsilon}(t),\xi^{\varepsilon}(t)>_{2,\rho}\Big|_{t=0}^{t=\tau}$$

$$- 2\int_0^{\tau}< \frac{\partial}{\partial t}\xi^{\varepsilon}(t), \frac{\partial}{\partial t}\xi^{\varepsilon}(t)>_{2,\rho}dt. \tag{2.15}$$

Now using (2.13) we may pass to the limit on the integral on the L.H.S. of (2.14), and using (2.11) and (2.12) we may pass to the limit in the terms of the R.H.S. Thus we obtain

$$\int_0^{\tau} 2< \frac{\partial^2}{\partial t^2}\xi(t),\varepsilon(t)>_{W_{\xi}^{*}\times W_{\xi}} = \frac{\partial}{\partial t}<\xi(t),\xi(t)>_{2,\rho}\Big|_{t=0}^{t=\tau}$$

$$- 2\int_0^{\tau} < \frac{\partial}{\partial t}\xi(t), \frac{\partial}{\partial t}\xi(t)>_{2,\rho}dt.$$

Again since all integrands, because of (2.12) to (2.14), are in $L^2[0,T]$ thus in $L^1(0,T]$, Lebesgue's differentiation theorem allows us to differentiate the integrals on both sides of (2.16) with respect to τ and recover the integrands, thus it follows that (2.10) holds and that $\frac{\partial}{\partial t}<\xi(t),\xi(t)>_{2,\rho}$ is an absolutely continuous function on $[0,T]$ being equal to the integral of an L^1 function.

We now return to show that (2.10) holds with $\xi^{\varepsilon}(t)$ replacing $\xi(t)$. For this purpose we use an embedding theorem of Lions and Magenes (1972) which reduces to the following special form needed later.

If $\eta(t)$ has the properties:

$$1)\qquad \eta(t) \in L^2([0,T] : W_\xi),$$

$$2)\qquad \frac{\partial \eta}{\partial t} \in L^2([0,T] : W_\xi), \tag{2.18}$$

$$3)\qquad \frac{\partial^2 \eta}{\partial t^2} \in L^2([0,T] : W_\xi^*),$$

then $\eta(t) \in A\,C([0,T] : W_\xi)$,

$$\frac{\partial \eta}{\partial t} \in AC([0,T] : H_\xi). \tag{2.19}$$

Moreover, if we let

$$X = \{u | u \in L^2([0,T] : W_\xi),\ \frac{\partial u}{\partial t} \subset L^2([0,T] : W_\xi),$$
$$\frac{\partial^2 u}{\partial t^2} \in L^2([0,T] : W_\xi^*)\ ,$$

with norm

$$\|u\|_X = \{\|u\|^2_{L^2([0,T]:W_\xi^*)} + \|\frac{\partial u}{\partial t}\|^2_{L^2([0,T] : W_\xi)}$$
$$+ \|\frac{\partial^2 u}{\partial t^2}\|^2_{L^2([0,T]:W_\xi^*)}\}^{1/2}$$

then

$$C_0^\infty([0,T] : W_\xi) \text{ is dense in } X.$$

Thus in particular $C^\infty([0,T] : W_\xi)$ is dense in X.

Given that $\xi^\varepsilon(t)$ satisfies the above conditions (cf. (2.12) to (2.14)), we may take $(\xi^\varepsilon)_n(t) \in (C^\infty([0,T) : W_\xi) \to \xi^\varepsilon(t)$ in X. Now starting with the relation

$$\int_0^\tau 2< \frac{\partial^2(\xi^\varepsilon)_n}{2t^2}(t),\ (\xi^\varepsilon)_n(t)>_{W_\xi^* \times W_\xi} dt = \frac{\partial}{\partial t}<(\xi^\varepsilon)_n(t),(\xi^\varepsilon)_n(t)>_{2,\rho}\Big|_{t=0}^{t=\tau}$$
$$- 2\int_0^t < \frac{\partial}{\partial t}(\xi_n^\varepsilon),\ \frac{\partial}{\partial t}(\xi^\varepsilon)_n>_{2,\rho}dt,$$

for $\tau \in [0,T]$, and using (2.17), we may pass to the limit in the above equation with the result

$$\int_0^\tau 2\langle \frac{\partial^2\xi^\varepsilon}{\partial t^2}(t),\ \xi^\varepsilon(t)\rangle_{W_\xi^*\times W_\xi} = \frac{d}{dt}\langle\xi^e(t),\xi^\varepsilon(t)\rangle_{2,\rho}\Big|_{t=0}^{t=\tau}$$

$$- 2\int_0^\tau \langle \frac{\partial}{\partial t}\xi^\varepsilon(t),\ \frac{\partial}{\partial t}\ \xi^\varepsilon(t)\rangle_{2,\rho} dt$$

for $t \in [0,\tau]$. So we obtain (2.15) as desired. This concludes the proof that (2.10) holds with $\xi^\varepsilon(t)$ replacing $\xi(t)$. We now invoke another lemma of Lions and Magenes (1972) which establishes that the energy equality

$$\|\frac{\partial\xi}{\partial t}\|^2_{H_\xi} + a(\xi,\breve{\xi}) = C \tag{2.20}$$

holds for solutions $(\xi,A)(t)$ to AEVP.

The proof now proceeds almost identically to that of Laval et. al. (1965). For completeness sake we present it here.

If $\xi(t)$ is a solution of the AEVP defined by (2.11), then

$$\langle \frac{\partial^2}{\partial t^2}\xi(t),\varepsilon(t)\rangle_{W_\xi^*\times W_\xi} + a_\xi(\xi,\xi) = 0 \tag{2.21}$$

holds because $\xi(t) \in W_\xi$.

If we make use of (2.9) and substitute for $a(\xi,\breve{\xi})$ from (2.19) into (2.21), we obtain the virial equation

$$4\|\frac{\partial\xi}{\partial t}(t)\|^2_{H_\xi} - \frac{d^2}{d^2t}\|\xi(t)\|^2_{H_\xi} = C. \tag{2.22}$$

We specify the initial data for AEVP to be

$$\xi\big|_{t=0} = \breve{\xi} \qquad \frac{\partial\xi}{\partial t}\Big|_{t=0} = \sigma\breve{\xi}\ . \tag{2.23}$$

Note that, with this choice of initial data, C in (2.22) is negative, so we obtain

$$\frac{d^2}{dt^2}\|\xi(t)\|^2_{2,\rho} > 4\|\frac{\partial\xi}{\partial t}\|^2_{2,\rho} \qquad \text{on} \qquad [0,T].$$

Thus the same argument as from (6.20) to (6.21), Part I, shows that

$$\|\xi(t)\|_{2,\rho}\frac{d^2}{dt^2}\|\xi(t)\|^2_{2,\rho} > \{\frac{d}{dt}\|\xi(t)\|_{2,\rho}\}^2 \quad \text{on} \quad [0,T] \tag{2.24}$$

Note also that, since according to (2.19) $\xi(t) \in AC([0,T] : H_\xi)$, and since $\|\xi(0)\| = \|\xi\| \neq 0$, if we choose $T' < T$ small enough. $\exists\sigma > 0$ such that

$$\|\xi(t)\|_{2,\rho} > \sigma \qquad \text{for} \quad t \in [0,T']. \tag{2.25}$$

Thus in particular if we define

$$y(t) = \log \frac{\|\xi(t)\|^2_{2,\rho}}{\|\xi\|^2_{2,\rho}},$$

$y(t)$ is absolutely continuous, since the logarithm is a Lipshitz function for any possitive interval bounded away from zero, and since the composition of a Lipshitz function with an absolutely continuous function is absolutely continuous (Stampachia, 1966).

Similarly,

$$\frac{dy}{dt} = \frac{d}{dt}\|\xi(t)\|^2_{2,\rho}/\|\xi(t)\|^2_{2,\rho},$$

is, because of (2.19), the quotient of an absolutely continuous function and an absolutely continuous function which is bounded away from zero, and hence (Rudin, 1974) is absolutely continuous.

Now (2.24) becomes

$$\frac{d^2y}{dt^2} > 0 \quad \text{on} \quad [0,T'),$$

with the initial conditions $y = 0$, $\frac{dy}{dt} = 2\sigma$. Thus, since $y(t)$ and $\frac{dy}{dt} \in Ac[0,T']$, we may integrate this inequality and obtain

$$\frac{dy}{dt} > 2\sigma, \quad \text{and} \quad y(t) > 2\sigma t \quad \text{for} \quad t \in [0,T'], \tag{2.25}$$

therefore,

$$\|\xi(t)\|^2_{2,\rho} > \|\xi(0)\|^2_{2,\rho} \exp 2\sigma t \quad \text{for} \quad t \in [0,T']. \tag{2.26}$$

We now note that the condition (2.25) is clearly satisfied for all $t > 0$ by repeating the same argument, we obtain

$$\|\xi(t)\|^2_{2,\rho} > \|\xi(0)\|^2_{2,\rho} \exp 2\sigma t.$$

This concludes our proof of Theorem 2.1.

§3. Variational Principle

Earlier we established that $a((\xi,A),(\xi,A))$ is coercive with respect to the W norm on $W \times W$. There exist $\exists\lambda > 0$ and $\delta > 0$ not necessarily unique such that

$$a((\xi,A),(\xi,A)) + \lambda\|\xi\|^2_{2,\rho} > \delta\|(\xi,A)\|^2_W, \qquad \forall(\xi,A) \in W.$$

In particular, λ may be chosen so that

$$a((\xi,A),(\xi,A)) + \lambda\|\xi\|^2_{2,\rho} > 0 \tag{3.1}$$

$$\forall(\xi,A) \in W. \tag{3.2}$$

We now show how to find the minimal λ, λ_0, that ensures (3.2) by a variational procedure.

We wish to find

$$\inf_{\substack{\|\xi\|^2_{2,\rho}=1 \\ (\xi,A)\in W}} a((\xi,A),(\xi,A)). \tag{3.3}$$

As seen from (2.1), this infimum is finite. We now show that, although in general there is no element $(\xi,A) \in W$ which realizes the infimum, we can construct a sequence of vectors $(\xi^n,A^n) \in W$ such that

$$a((\xi^n,A^n,(\xi^n,A^n)) = \lambda_n + \lambda_0 .$$

We use again the skew orthonormal basis

$$\beta^i = (\beta^i_1,\beta^i_2), \qquad < \beta^i_1,\beta^j_1 >_{2,\rho} = \sigma_{ij},$$

and define $\hat{\Lambda}^n = (\hat{\xi}^n,\hat{A}^n)$ to be an element of W of the form

$$\Lambda^n = (\xi^n,A^n) = \sum_{i=1}^{n} \alpha_{in}(\beta^i_1,\beta^i_2), \tag{3.4}$$

with $\|\xi^n\|_{2,p} = 1$, for which the infimum of (3.3) is attained. That this infimum is indeed reaalized by an element of W of the form (3.4) follows by noting that

$$\xi^n = \sum_{i=1}^{n} \alpha_{in} \beta_1^i ,$$

$$\|\xi^n\|^2 = \sum \alpha_{in}^2 ,$$

so that the α_{in} vary over a compact set of dimension n. Since $a((\xi^n,\Lambda^n),(\xi^n,\Lambda^n)$ is a continuous function of α_{in}, it attains both its maximum and minimum and the minimum satisfies

$$a(\hat{\Lambda}^n,\beta^i) = \lambda^n(\hat{\xi}^n,\beta_1^i), \qquad 1 \le i \le n, \tag{3.5}$$

$$\lambda^n = a(\hat{\Lambda}^n,\hat{\Lambda}^n). \tag{3.6}$$

The equation (3.5) holds because $\forall r > 0$, $\forall\Sigma$ of the form (3.4), $\Sigma = (\Sigma_1,\Sigma_2)$ such that

$$a(\hat{\Lambda}^n + r\Sigma,\hat{\Lambda}^n + r\Sigma) \ge \lambda^n\langle\xi^n + r\Sigma_1,\xi^n + \Sigma_1\rangle_{2,\rho}$$

$$a(\hat{\Lambda}^n,\hat{\Lambda}^n) + r^2a(\Sigma,\Sigma) + 2ra(\hat{\Lambda}^n,\Sigma) \ge \lambda^n\langle\hat{\xi}^n,\hat{\xi}^n\rangle_{2,\rho} + \lambda^n r^2\langle\Sigma_1,\Sigma_2\rangle_{2,\rho}$$
$$+ 2\lambda^n r\langle\Sigma_1,\hat{\xi}\rangle_{2,\rho}.$$

Using

$$a(\hat{\Lambda}^n,\hat{\Lambda}^n) = \lambda^n\langle\xi^n,\xi^n\rangle = \lambda^n ,$$

to cancel one term from both sides and then dividing by r and letting $r \to 0$, we have

$$a(\hat{\Lambda}^n,\Sigma) \le \lambda^n\langle\xi^n,\Sigma_1\rangle_{2,\rho}.$$

Therefore

$$a(\hat{\Lambda}^n,\Sigma) \ge \lambda^n\langle\xi^n,\Sigma_1\rangle_{2,\rho}$$

follows from replacing Σ by $-\Sigma$ in the above derivation. Now λ^n is an increasing sequence of real numbers since the infimum is taken over an increasing sequence of sets. Moreover, by the coerciveness there exist λ such that

$$a(\hat{\Lambda}^n,\hat{\Lambda}^n) + \lambda\|\hat{\xi}^n\|_{2,\rho} \ge 0,$$

that is,

$$\lambda_n \geq -\lambda.$$

Hence the sequence of $\{\lambda^n\}$ is bounded below and there exists λ_0 such that

$$\lambda^n \downarrow \lambda_0,$$

where λ_0 may be negative. Moreover, by the coerciveness again, $\delta, \sigma^2 > 0$ such that $\sigma^2 > -\lambda_0$,

$$a((\xi,A),(\xi,A)) + \sigma^2 \geq \delta\|(\xi,A)\|_W^2 , \tag{3.5}$$

where $\|\xi\|_{2,\rho}^2 = 1$. It follows that

$$\hat{\Lambda}^n = (\hat{\xi}^n,\hat{A}^n),$$

forms a bounded sequence in W, since

$$a((\xi^n,A^n),(\xi^n,A^n)) + \sigma^2 \geq \lambda_0 + \sigma^2 \geq \delta\|(\hat{\xi}^n,\hat{A}^n)\|_W, \qquad \forall n.$$

Hence there exists a weakly convergent subsequence in W of the $\hat{\Lambda}^n$, and in particular, of the $\hat{\xi}^n$ in $L_\rho^2(\Omega_p)$. Unfortunately, without some additional features of W ensuring compactness of the embedding of W in $L^2(\Omega_p) \times L^2(\Omega_v)$, we may not conclude that the limit $\hat{\Lambda}$ is a genuine eigenfunction. We will not examine this question further here. In any case we have shown

$$a(\hat{\Lambda}^n,\hat{\Lambda}^n) = \lambda^n \downarrow +\lambda_0$$

where $\hat{\Lambda}^n$ is the solution to the n^{th} minimization problem discussed above. Finally we have

<u>Theorem 3.1.</u>

$$\inf_{\substack{\|\xi\|_{2,\rho}^2=1 \\ (\xi,A)\in W}} a((\xi,A),(\xi,A)) = \lambda_0 .$$

Proof:

$$\inf_{\substack{\|\xi\|_{2,\rho}^2=1 \\ (\xi,A)\in W}} a((\xi,A),(\xi,A)) \leq \lambda_0 ,$$

is clear since $(\hat{\xi}^n n\hat{A}^n)$ is a collection of elements of W with $\|\hat{\xi}^n\|_{2,\rho} = 1$.

$$\inf_{\substack{\|\xi\|^2_{2,\rho}=1 \\ (\xi,\Lambda)\in W}} ((\xi,\Lambda),(\xi,\Lambda)) > \lambda_0 ,$$

is shown by contradiction.

If

$$\inf_{\substack{\|\xi^2\|^2_{2,\rho}=1 \\ (\xi,\Lambda)\in W}} a((\xi,\Lambda),(\xi,\Lambda)) < \lambda_0 - \varepsilon$$

then there exists $N > 0$ such that for $n > N$

$$a(\hat{\Lambda}^n,\hat{\Lambda}^n) < \lambda_0 - \frac{\varepsilon}{2} .$$

But this contradicts the defining property of $\hat{\Lambda}^n$ and the monotonicity of the sequence λ^n.

A consequence of (2.7) is the following:

Corollary:

$$a((\xi,\Lambda),(\xi,\Lambda)) - \lambda_0\langle\xi,\xi\rangle_{2,\rho} > 0$$
$$\forall(\xi,\Lambda) \in W.$$

Proof:

Since

$$\inf_{\substack{\|\xi^2\|_{2,\rho}=1 \\ (\xi,\Lambda)\in W}} a((\xi,\Lambda),(\xi,\Lambda)) = \lambda_0 ,$$

$$a((\xi,\Lambda),(\xi,\Lambda)) - \lambda_0 > 0.$$
$$\forall(\xi,\Lambda) \in W \text{ with } \|\xi\|^2_{2,\rho} = 1.$$

Now for arbitrary $(\xi,\Lambda) \in W$, consider

$$(\tilde{\xi},\tilde{\Lambda}) = (\xi,\Lambda)/\|\xi\|^2_{2,\rho} ,$$

where $\|\tilde{\xi}\|_{L^2_\rho(\Omega_p)} = 1$. Then

$$a((\tilde{\xi},\tilde{\Lambda}),(\tilde{\xi},\tilde{\Lambda})) - \lambda_0 > 0.$$

However, due to the bilinearity of $a(\cdot,\cdot)$

$$a((\tilde{\xi},\tilde{\Lambda}),(\tilde{\xi},\tilde{\Lambda})) = a((\xi,\Lambda),(\xi,\Lambda))/\|\xi\|^2_{2,\rho}.$$

We obtain (2.9) by multiplying (2.10) by $\|\xi\|^2_{2,\rho}$.

§4. Estimate of the Maximum Growth Rate.

In deriving the sufficiency of the modified energy principle in Theorem 2.1, we established that if $\exists\sigma^2$ such that

$$a((\xi,A),(\xi,A)) + \sigma^2\|\xi\|^2_{2,\rho} > 0, \qquad \forall(\xi,A) \in W,$$

then any solution $(\xi(t),A(t))$ of the EVP on $[0,\infty)$ is subject to the growth estimate

$$\|\dot{\xi}\|^2_{2,\rho} \leq c \exp 2\sigma t \quad \text{on} \quad [0,\infty). \tag{4.1}$$

In the process of proving the coerciveness of the bilinear form $a((\xi,A)(\xi,A))$ (§5, Part I), that is, $\exists\lambda,\delta > 0$ such that

$$a((\xi,A),(\xi,A)) + \lambda\|\xi\|^2_{2,\rho} > \delta\|(\xi,A)\|^2_W, \qquad \forall(\xi,A) \in W$$

we found that upper bounds on λ could be given in terms of the constants e_i, which characterize the equilibrium quantities p, B, $\nabla \times B$. However, in §5, Part I, δ on the R.H. S. of equation (5.16) was taken to be $1/2$. We now make note of what estimate is obtained for λ when $\delta = 0$, since this is all that is needed in order to obtain the growth estimate (4.1).

Thus we let f_1 and f_2 in (5.15), Part I be such that

$$e_1 f_1 = 1 \qquad e_3 e_4 f_2 = 1$$

then

$$\lambda = \frac{e_1^2 e_2}{4} + \frac{e_2 e_3^2 e_4}{4}$$

is an upper bound for the maximum growth rate; or we make use of the more precise estimate following from (5.13) and (5.14) in PartI,

$$\lambda = \left\| \frac{2\nabla \times B \times n \cdot B \cdot \nabla n}{\delta} \right\|_{L^\infty},$$

is, again, an upper bound on the maximum growth rate.

§5. Diffuse Pinch and Multi-Tori Configuration.

In this section we extend the results of §3, Part I to two other configurations of importance in thermonuclear research. One case deals with a diffuse pinch and the other, a multiple tori plasma confined in a toroidal shell.

In a diffuse pinch (Figure 1) the plasma extends to the confining shell. The equations in the plasma region are the same as in §3, Part I; now we only need to express them in Eulerian coordinates

$$\rho\ddot{\xi} = \nabla(\gamma p \nabla \cdot \xi + \xi \cdot \nabla p) + \nabla \times B \times \nabla \times (\xi \times B)) \tag{5.1}$$

$$+ \nabla \times \nabla \times (\xi \times B) \times B.$$

We impose the boundary condition on the confining shell that the normal component of velocity on the shell is zero. By integration

$$\xi \cdot n = 0 \quad \text{on} \quad \Gamma, \tag{5.2}$$

and we note that if (5.2) is satisfied then the perturbed magnetic field

$$B^1 = \nabla \times (\xi \times B)$$

automatically satisfies the boundary condition:

$$B^1 \cdot n = 0 \quad \text{on} \quad \Gamma,$$

because

$$\xi \cdot n = 0 \Longrightarrow n \times (\xi \times B) = 0 \quad \text{on} \quad \Gamma,$$

where $B \cdot n = 0$ by our assumption on the equilibrium state, and

$$n \times v = 0 \Longrightarrow n \cdot \nabla \times v = 0 \quad \text{on} \quad \Gamma,$$

as can be easily seen by applying Stoke's theorem to infinitesimal loops on the boundary.

We now present, in a somewhat condensed fashion, an existence and uniqueness proof for a weak solution to the boundary value problem (5.1) to (5.2) with initial conditions

$$\xi(0) = \xi_0 \,, \quad \frac{\partial \xi}{\partial T}(0) = \dot{\xi}_0. \tag{5.3}$$

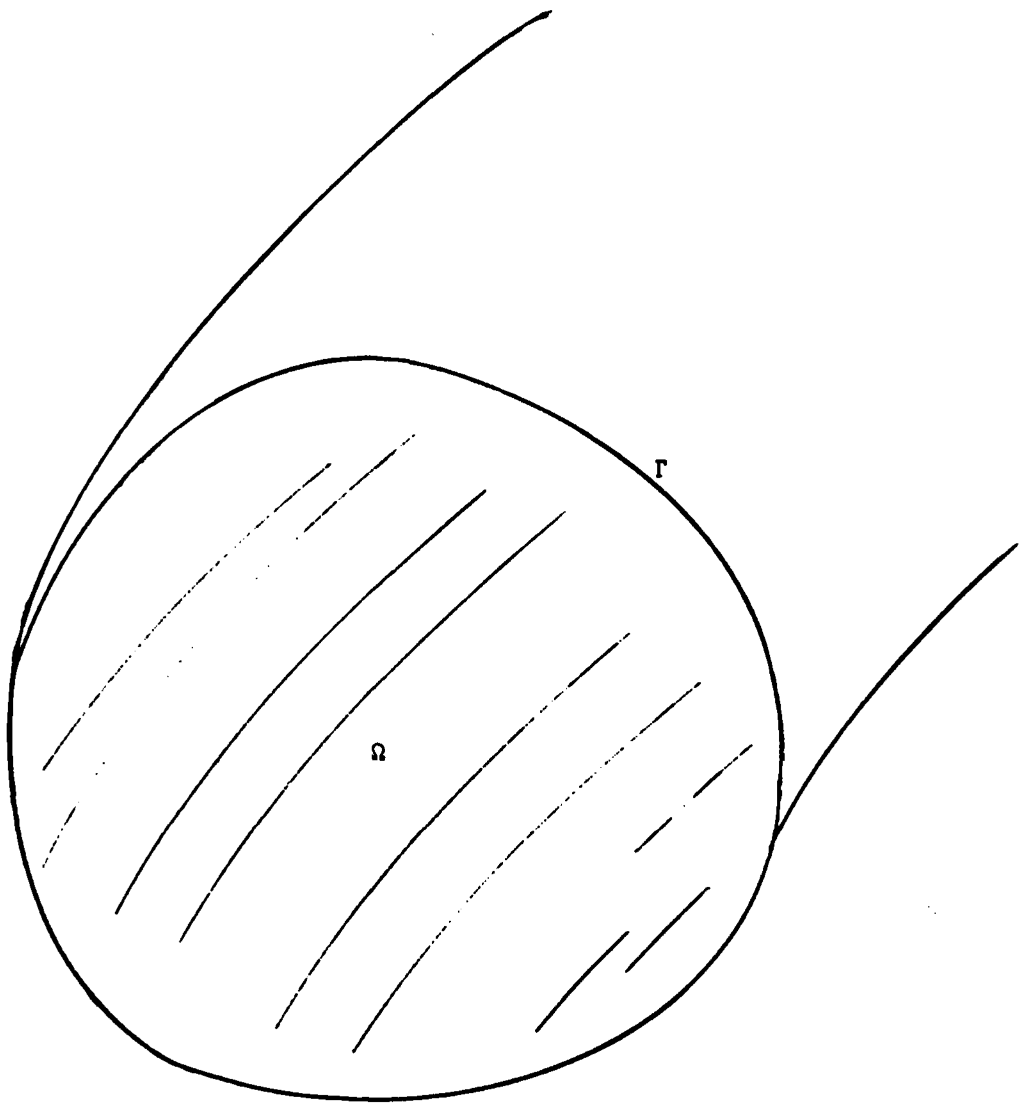

Figure 1

A Cross Section of the Diffuse Pinch

We shall make the following assumptions on the equilibrium quantities:

$$p,\ \nabla p \in L^{\infty},$$

$$B \cdot n = 0 \quad \text{on} \quad \Gamma.$$

Furthermore, we also assume $\exists c_1, c_2, d_1, d_2 > 0$ such that

$$c_1 < p(x) < c_2\ ,$$

$$d_1 < \rho(x) < d_2.$$

If we take the inner product of (5.1) with $\breve{\xi}$ where $\breve{\xi}$ satisfies the boundary condition $\breve{\xi} \cdot n = 0$ on Γ, we get

$$\langle \rho\ddot{\xi}, \xi \rangle_{2,\rho} = - \int_{\Omega_P} (\gamma P \nabla \cdot \xi \nabla \cdot \breve{\xi} + \nabla \times (\xi \times B) \cdot \nabla \times (\breve{\xi} \times B)$$

$$- \breve{\xi} \cdot \nabla \times B \times (\nabla \times (\xi \times B)) + (\nabla \cdot \breve{\xi})(\xi \cdot \nabla P) dv$$

$$\pm \int_{\Gamma_P} (\breve{\xi} \cdot n)[\gamma P \nabla \xi - B \cdot (\nabla \times \xi \times B) - \xi \cdot \nabla P] dS$$

$$= - \int_{\Gamma_P} \gamma P (\nabla \cdot \xi)(\nabla \cdot \breve{\xi}) + \nabla \times (\xi \times B) \cdot \nabla \times (\breve{\xi} \times B)$$

$$- \breve{\xi} \cdot \nabla \times B \times (\nabla \times (\xi \times B)) + (\nabla \cdot \breve{\xi})(\xi \cdot \nabla P) dv.$$

Thus we define

$$a((\xi, \breve{\xi}) = \int_{\Omega_P} \gamma P (\nabla \cdot \xi)(\nabla \cdot \breve{\xi}) + \nabla \times (\xi \times B) \cdot \nabla \times (\breve{\xi} \times B)$$

$$- \breve{\xi} \cdot \nabla \times B \times (\nabla \times (\xi \times B)) + (\nabla \cdot \breve{\xi})(\xi \cdot \nabla p) dv, \tag{5.4}$$

and obtain

$$\langle \rho\ddot{\xi}, \breve{\xi} \rangle_{2,P} + a(\xi, \breve{\xi}) = 0 \quad \forall \breve{\xi} \ \text{such that} \ \breve{\xi} \cdot n = 0 \ \text{on} \ \Gamma.$$

Let

$$V^{-} = \{\xi \in H^1(\Omega) \,|\, (\xi \cdot n) = 0\},$$

with scalar product

$$\langle \xi, \breve{\xi} \rangle_{V^{-}} = \int_{\Omega_P} (\gamma_P (\nabla \cdot \xi)(\nabla \cdot \breve{\xi}) \tag{5.5}$$

$$+ \nabla \times (\xi \times B) \cdot \nabla \times (\breve{\xi} \times B) + \rho \xi \cdot \breve{\xi}) dv,$$

and let

$$V = \text{closure of } V^{-} \text{ in } \| \|_{V^{-}} .$$

All the properties related to volume integrals enjoyed by the first components of elements of W (cf. §4, Part I) also hold for elements of V and the proofs are the same, that is

$$\nabla \cdot \xi \in L^2(\Omega),$$

$$\nabla \times (\xi \times B) \in L^2(\Omega),$$

$$\xi \in L^2(\Omega).$$

We now make use of a result in Temam (1973), noting that given (5.5), $\exists c > 0$ such that,

$$\|\xi\|_V > c\|\xi\|_{E(\Omega)} ,$$

so that if ξ^m is a Cauchy sequence in V^-, with

$$\xi^m \to \xi \quad \text{in} \quad V,$$

then

$$n \cdot \xi^m \to n \cdot \xi \quad \text{in} \quad H^{-1/2}(\Gamma).$$

Therefore, since $n \cdot \xi^m = 0$ in $H^{-1/2}(\Gamma)$ (in fact in $H^{1/2}(\Gamma)$)),

$n \cdot \xi = 0$ in $H^{-1/2}(\Gamma)$.

All other aspects of the existence and uniqueness proofs are straightforward simplifications of those in §6, Part I, so we will not repeat them here but only state the following

__Theorem 5.1:__

There exist a unique solution $\xi(t)$ of the EVP

(1) $$\frac{\partial^2}{\partial^2 t} \langle \xi(t), \tilde{\xi} \rangle_{2,\rho} + a(\xi, \tilde{\xi}) = 0 \quad \forall \tilde{\xi} \in V,$$

(2) $$\xi(0) = \xi_0 \in V, \quad \frac{\partial \xi}{\partial t}(0) = \dot{\xi}_0 \in L^2(\Omega),$$

and

$$\xi(t) \in C^0([0,\infty) : V),$$

$$\frac{\partial \xi}{\partial t} \in C^0([0,\infty) : L^2(\Omega)).$$

One can easily verify that for the diffuse pinch the estimate

$$\|\dot{\xi}(t)\|^2_{2,\rho} < ((1 + \frac{C_1}{\delta^2}\|\xi(0)\|^2_{2,\rho} + \|\dot{\xi}(0)\|^2)e^{2\sigma t},$$

holds, where

σ^2 can be estimated from (5.13), Part I,

$$\sigma^2 = \left\| \frac{2\rho \times B \times n \cdot B \cdot \nabla_n}{\rho} \right\|_{L^\infty}.$$

If the plasma region, instead of consisting of one torus, is the union of N non-intersecting tori with interior Ωp_i, all enclosed in an outer toroidal conducting shell (Figure 2) Γ_v, the basic approach of Part I again applies.

In each connected plasma domain Ωp^i we search for $\xi^i(t)$ such that

$$\rho_i \ddot{\xi}^i(t) = F(\xi^i) = \nabla(\gamma p \nabla \cdot \xi^i + \xi^i \cdot \nabla p^i) \tag{5.6}$$

$$+ \nabla \times B^i \times \nabla \times (\xi^i \times B^i) + \nabla \times \nabla \times (\xi^i \times B^i) \times B^i.$$

In the vacuum region we gain require

$$\nabla \times \nabla \times A = 0,$$

$$\nabla \times A = 0.$$

In each plasma-vacuum interface Γ_{p_i} the continuity of the total pressure requires that

$$- \gamma p^i \nabla \cdot \xi^i + B_0^{p^i}(\xi^i \cdot \nabla B^{p^i} + \nabla \times (\xi^i \times B^{p^i})) \tag{5.9}$$

$$= B^v \cdot (\nabla \times A + \xi^i \cdot \nabla B^v), \quad \text{on } \Gamma_{p^i}.$$

Furthermore,

$$n \times A = 0 \quad \text{on } \Gamma_v. \tag{5.10}$$

We also have the prescribed flux

$$\int_{\Gamma_v} A \cdot n \, ds = \delta(t), \tag{5.11}$$

and initial conditions

$$\xi(0) = \xi_0, \quad \frac{\partial \xi}{\partial t}(0) = \dot{\xi}_0.$$

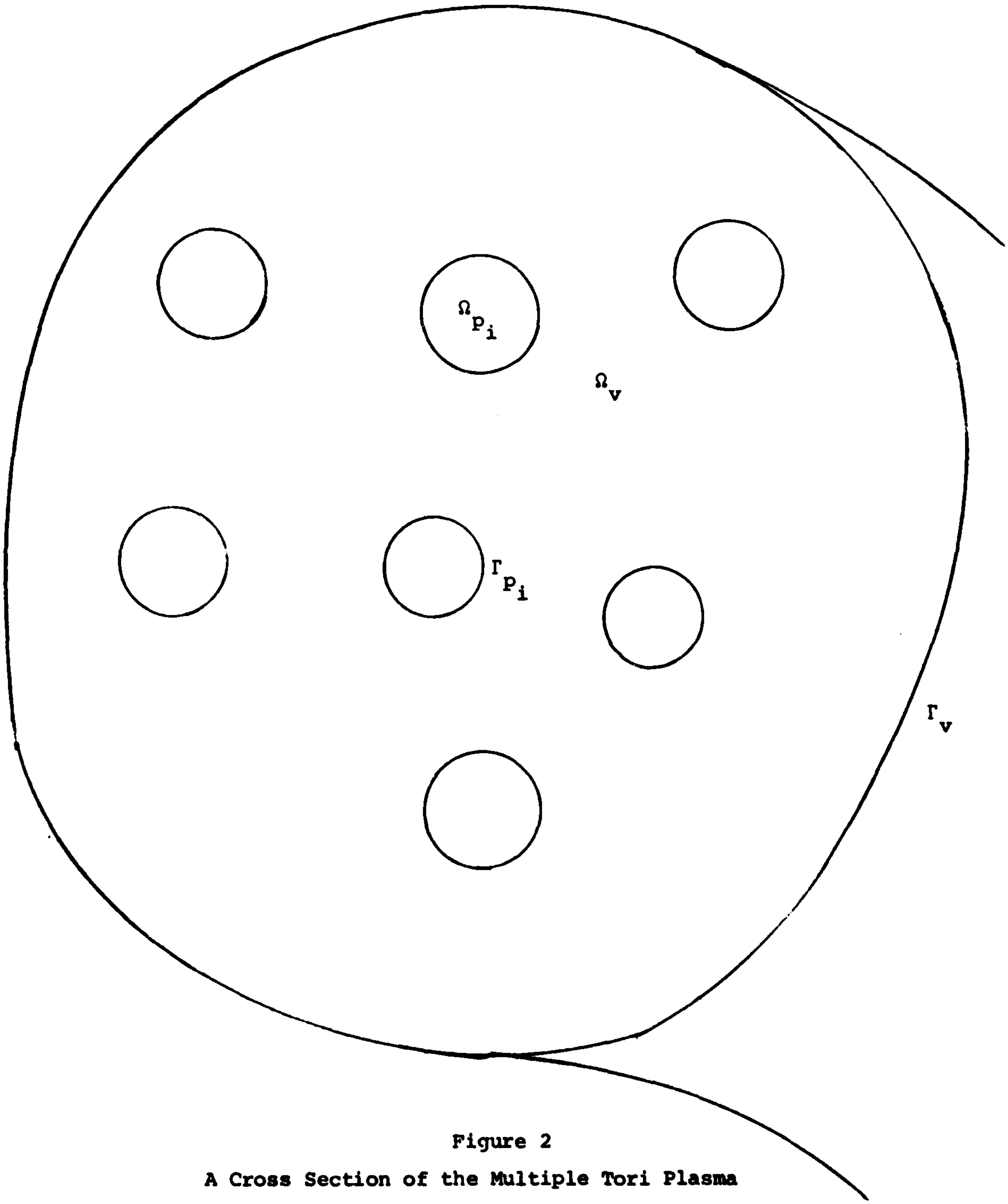

Figure 2

A Cross Section of the Multiple Tori Plasma

Proceeding in the same way as in Part I, we define a space

$$W^- = \{(\xi^1,\cdots,\xi^{N-1},\xi^N) | \xi_i \in H^1(\Omega p_i),\quad A \in \bar{V}_2(\Omega_v) \quad H^1(\Omega_v)\}$$

$$n \times A = 0 \quad \text{on} \quad \Gamma_v ,$$

$$n \times A = (-n \cdot \xi^i)B^v \quad \text{on} \quad \Gamma_{p^i},$$

with $\|(\xi,A)\|_{W^-}$ defined by

$$(\xi^1,\xi^2,\cdots,\xi^N,A)\|_{W^-} = \int_{\Omega_v} |\nabla \times A|^2 dv + \sum_i \int_{\Omega p_i} \{\gamma P^i(\nabla \cdot \xi_i^i)^2$$
$$+ |\nabla \times (\xi^i \times B^i)|^2 + \rho \xi_i^i\} dv,$$

and let

$$W = \text{closure of } W^- \text{ in } \| \ \|_{W^-}$$
$$W^+ = W \oplus (0,A'),\quad A' \in \bar{V}_3 .$$

We note that in order to determine a unique solution to the elliptic system:

$$\nabla \times \nabla \times A = 0,$$
$$\nabla \cdot A = 0,$$
$$n \times A = 0, \quad \text{on} \quad [\bigcup_{i=1}^{n-1} \Gamma_{p_i}] \cup \Gamma_v ,$$

it is now necessary to prescribe [Blank, Grad, Fredrichs, 1957] $N - 1$ fluxes

$$\int_{\Gamma p_i} A \cdot n \, ds = \delta_i(t) ,$$

on any $N - 1$ of the N plasma surfaces say $\Gamma_{p_1},\cdots,\Gamma_{p_{N-1}}$, in addition to (5.11).

With conditions on the equilibrium quantities identical to those of §3, Part I, and the same procedures as in §3 to §6, Part I, we may again establish the existence and uniqueness of solutions to the following AEVP associated with (5.6)-(5.11):

(1) $$\frac{\partial^2}{\partial t^2}\langle \xi_1,\cdots,\xi^N),(\tilde{\xi}_1,\cdots,\tilde{\xi}^N)\rangle_{2,\rho}$$
$$+\ a(((\xi^1,\cdots,\xi^N),\hat{A}),(\tilde{\xi}^1,\cdots,\tilde{\xi}^N),\tilde{\hat{A}})) = 0 \qquad (5.12)$$
$$\forall((\tilde{\xi}^1,\cdots,\tilde{\xi}^N),A) \in W.$$

(2) $$(\xi^1(0),\cdots,\xi^N(0),\hat{A}(0)) = (\xi_0^1,\cdots,\xi_0^N,A_0 - A_0') \ C\text{-}W$$
$$(\frac{\partial \xi^1}{\partial t}(0),\cdots,\frac{\partial \xi^N}{\partial t}(0)) = (\dot{\xi}^1,\cdots,\dot{\xi}^N)_0 \in H, \qquad (5.13)$$

where $H = \{\text{closure in } L^2 \text{ of } P_{\Omega P}(W)\}$, and A_0' is a solution of (6.13), corresponding to fluxes $\delta(0)$ and $\delta_j(0)$.

(3) $$\int_{\Gamma_v} \hat{A}\cdot n\,ds = 0, \quad \int_{\Gamma_{P_i}} \hat{A}\cdot n\,ds = 0. \qquad (5.14)$$

<u>Theorem 5.2:</u>

There exists a unique solution of the AEVP (5.12-5.14) such that
$$(\xi^1(t),\cdots,\xi^N(t),\hat{A}(t)) \in W,$$
$$(\frac{\partial \xi^1}{\partial t}(t),\cdots,\frac{\partial \xi^N}{\partial t}(t)) \in C^0([0,\infty) : H).$$

To the solution of AEVP we add a solution of the elliptic system
$$\nabla\times\nabla\times A' = 0,$$
$$\nabla\cdot A' = 0,$$
$$n\times A' = 0 \text{ on } \bigcup_i \Gamma P_i \cup \Gamma_v,$$
$$\int_{\Gamma_v} A'\cdot n\,ds = \delta(t), \quad \int_{\Gamma_{P_i}} A'\cdot n\,ds = \delta j(t),$$
where $A' \in \overline{V}_3$.

As before this solution has the property that

(1) $$\frac{\partial^2}{\partial t^2}\langle(\xi^1,\cdots,\xi^N)(t),(\xi^1,\cdots,\xi^N)\rangle_{2,\rho}$$
$$+\ a((\xi^1,\cdots,\xi^N)(t),A(t)(\xi^1,\cdots,\xi^N(t),A(t)) = 0$$
$$\forall(\tilde{\xi}^1,\cdots,\tilde{\xi}^N(t),\tilde{A}(t)) \in W^+.$$

where

$$a((\xi^1,\cdots,\xi^N,A),(\tilde{\xi}^1,\cdots,\tilde{\xi}^N,\tilde{A})) = \sum_{i=1}^{i=N} a((\xi^i,A^i),(\tilde{\xi}^i,\tilde{A}^i)).$$

(2) $$((\xi^1(0),\cdots,\xi^N(0)),A(0)) = ((\xi_0^1,\cdots,\xi_0^N),A_0) \in W^+$$

$$\left(\frac{\partial \xi^1}{\partial t}(0),\cdots,\frac{\partial \xi^N}{\partial t}(0)\right) = (\dot{\xi}_0^1,\cdots,\dot{\xi}_0^N) \in H.$$

(3) $$\int_{\Gamma_v} A \cdot n \, ds = \sigma(t); \quad \int_{\Gamma P_{ij}} A \cdot n \, ds = \sigma_j(t).$$

If $\sigma(t)$ is continuous, then

$$((\xi^1(t),\cdots,\xi^N(t)),A(t)) \in C([0,T] : W^+)$$

$$\left(\frac{\partial \xi^1}{\partial t},\cdots,\frac{\partial \xi^N}{\partial t}\right) \in C^0([0,T] : H).$$

Here

$$\|(\xi^1,\cdots,\xi^N)\|_{W^+}^2 = \|(\xi^1,\cdots,\xi^N)\|_W^2 + \|A\|_{2,v}^2 .$$

It can be shown that growth estimates of the form

$$\|\dot{\xi}^i\|_{2,\rho}^2 \leq C \exp 2\sigma t$$

again hold, where σ^2 is the smallest positive real number such that

$$a((\xi^1,\cdots,\xi^N,A),(\xi^1,\cdots,\xi^N,A)) + \sigma^2 \sum_{i=1}^{i=N} \|\xi^i\|_{2,\rho}^2$$
$$\geq \delta\|(\xi^1,\cdots,\xi^N),A)\|_W^2.$$

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Based upon the existence and uniqueness of a solution to the linearized Lundquist equations established previously, the modified energy principle for the σ-stability of a confined toroidal plasma is rigorously justified. A variational principle is developed to find an infimum of σ, and an estimate for the maximum growth rate is obtained. The results are also extended to a diffuse pinch and a multiple tori plasma.

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